9. POLYNOMIALS

§9.1. Definition of a Polynomial

A **polynomial** is an expression of the form:

$$\mathrm{a}(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0.$$

The symbol '*x*' is called an **indeterminate** and simply plays the role of a place marker. The role of the '*x*' is to provide positions in the expression which can be replaced (substituted) by a value. The numbers a_0, a_1,... are called the **coefficients** of the polynomial.

Example 1: The expression $\mathrm{a}(x) = x^3 - 4x^2 + 7x - 11$ is a polynomial in *x*. The coefficients of a(x) are the numbers 1, –4, 7, –11.

The powers of *x* in a polynomial must be non-negative integers and there must be a finite number of terms. Other expressions have the appearance of being polynomials but, because some powers are negative or fractional, or because there are infinitely many terms, they are not considered to be polynomials.

Example 2: The expressions

$$x + \frac{1}{x}$$

$$1 - \sqrt{x} + x^2$$

$$\text{and } 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

are not polynomials even though they are built up from powers of x.

The coefficients can be rational numbers, forming the set $\mathbb{Q}$, real numbers, forming the set $\mathbb{R}$, or complex numbers, forming the set $\mathbb{C}$. These systems, called **fields**, have the property that x^{-1} exists for every non-zero x. (This somewhat loose description will do for our present purposes. An exact definition comprises 11 separate properties, or axioms.)

Sometimes we shall restrict our attention to polynomials with integer coefficients even though the integers don't form a field. Notice that a rational polynomial can always be converted to an integer polynomial by multiplying by a common denominator.

Example 3: The rational polynomial $\frac{22}{7}x^3 - \frac{5}{2}x + \frac{3}{4}$ is equal to $\frac{88x^3 - 70x + 21}{28}$.

It is also possible to consider polynomials whose coefficients are integers modulo p for some prime number, p. These systems are fields and the theory of polynomials over these finite fields works nicely even if some of the results look a bit strange.

For example over $\mathbb{Z}_2$, the field of integers mod 2 (where there are just two elements, 0 and 1 with $1 + 1=0$), the factorisation of $x^2 + 1 = (x + 1)^2$, since the $2x$ term is equal to 0.

§9.2. Degree of a Polynomial

The **degree** of a polynomial is the largest power of x that occurs with a non-zero coefficient. That is, if $a(x)=a_nx^n + a_{n-1}x^{n-1} + ... + a_1x + a_0$, the degree of $a(x)$ is n (provided that $a_n \neq 0$). We can write **deg** $a(x) = n$.

Example 4: Polynomials of degree 2 are **quadratics**, of the form $ax^2 + bx+c$ (where $a \neq 0$).
Polynomials of degree 1 are the so-called **linear polynomials** such as $2x + 3$ and $½x – ¼$.
Polynomials of degree 0 are the non-zero **constant polynomials**. It might seem strange for numbers such as 3 or –½ to be considered as polynomials, but they can be. If it makes you feel better you can write 3 as $0x^2 + 0x + 3$, but its degree vwould still be zero. Remember that perhaps you once found it strange to call 3 a fraction until you learnt that it could be written as $\frac{3}{1}$.

There's one polynomial for which the degree remains undefined. It's the zero constant polynomial, 0. (Some books define this degree to be $-\infty$ for certain technical reasons but we'll leave it undefined.)

The coefficient of x^n, for a polynomial of degree n, is called its **leading coefficient**. For example the leading coefficient of $3x^2 - x + 5$ is 3. But beware. The leading coefficient doesn't always come first — the leading coefficient of $1 - x^2$ is -1, not 1.

A **monic** polynomial is one where the leading coefficient is 1. Clearly every non-zero polynomial can be made monic by dividing it by its leading coefficient.

Example 5: The polynomial $4x^3 - 8x + 1$ has degree 3. Its leading coefficient is 4 and so it's not monic. However it can be expressed as 4 times the monic polynomial $x^3 - 2x + ¼$.

If F is a field (for example F might be $\mathbb{Q}$, the field of rational numbers) we denote the set of all polynomials with coefficients coming from F by the symbol **F[*x*]**.

Example 6: For example $\mathbb{R}[x]$ contains the polynomial $\pi x^2 - 2x + \sqrt{3}$ but not $x^2 + i$. However $x^2 + i$ is in $\mathbb{C}[x]$.

§9.3. Addition and Multiplication of Polynomials

Polynomials are added (and subtracted) in the usual way – just add (or subtract) the corresponding coefficients. Multiplication is somewhat more complicated to describe abstractly, yet it just amounts to expanding the product of the expressions in the way you have always done. For example $(ax^2 + bx + c)(dx + e) = adx^3 + (ae + bd)x^2 + (be + cd)x + ce$.

With these operations the system F[x] behaves very much like a field itself, but with one very important difference. In a field nearly every number has an inverse under multiplication (in fact 0 is the only exception). Most polynomials, on the other hand, do *not* have inverses. For example, since $\frac{1}{x}$ and $\frac{1}{1-x}$ are not polynomials, the polynomials x and $1 - x$ don't have (polynomial) inverses. In fact the *only* polynomials with inverses are the non-zero constant polynomials such as –2 (whose inverse is the constant polynomial –½).

Now it *is* possible to write $\frac{1}{1-x}$ as $1 + x + x^2 + \ldots..$ but although this looks like a polynomial it has infinitely many terms while a polynomial, by definition, has only finitely many. An expression like $1 + x + x^2 + \ldots$ is called a **power series**.

The system F[x] of polynomials over a field, in fact, behaves much more like the system of integers (where only ±1 have integer inverses).

Theorem 1: For polynomials a(x), b(x) $\in$ F[x], where F is a field:

$$\deg [a(x)b(x)] = \deg a(x) + \deg b(x)$$

Proof: This follows from the fact that $(a_m x^m + \ldots)(b_n x^n + \ldots) = a_m b_n x^{m+n} + \ldots$, and the fact that if a_m and b_n are non zero then $a_m b_n$ is also non-zero. ✋☺

Thus the degree of a product is the *sum* of the degrees. Does this remind you of something? The degree function behaves like the logarithm function.

Theorem 2: For polynomials a(x), b(x) $\in$ F[x]

$$\deg [a(x) + b(x)] \le \max[\deg a(x), \deg b(x)).$$

There is nothing really surprising in this result except for the question as to why "less-than-or-equals" rather than just "equals". The answer is that when we add two polynomials of the same degree the leading coefficients can cancel thereby producing a polynomial of lower degree.

Example 7: If $a(x) = 2x^2 - x + 7$ and $b(x) = -2x^2 + 4x + 1$ then $a(x) + b(x) = 3x + 8$, which has *smaller* degree than either a(x) or b(x).

§9.4. Division and Remainders

As mentioned earlier, one polynomial doesn't usually divide exactly into another. Like the system of integers we're usually left with a **remainder**. We get exact divisibility precisely when the remainder is zero. Furthermore this remainder is in some sense *smaller* than whatever we're dividing by. For polynomials, *smaller* means 'of smaller degree'.

The process of obtaining the remainder on dividing one polynomial by another is very similar to the familiar long-division algorithm.

Example 8:

$$
\begin{array}{r|l}
 & 2x + 4 \\
\hline
x^2 - 2x + 7 & 2x^3 \qquad + 5x - 3 \\
 & \underline{2x^3 - 4x^2 + 14x} \\
 & \qquad 4x^2 - 9x - 3 \\
 & \qquad \underline{4x^2 - 8x + 28} \\
 & \qquad\qquad - x - 31
\end{array}
$$

From this calculation we conclude that the remainder on dividing $2x^3 + 5x - 3$ by $x^2 - 2x + 7$ is $-x - 31$. Note that the remainder has lower degree than $x^2 - 2x + 7$, the polynomial we are dividing by. Note also how we write the terms neatly underneath others of the same degree.
The result of the calculation can also be expressed as:
$2x^3 + 5x - 3 = (x^2 - 2x + 7)(2x + 4) + (-x - 31)$.

Theorem 3 (Division Algorithm):
If $a(x)$, $b(x)$ are polynomials and $b(x)$ is non-zero then $a(x) = b(x)q(x) + r(x)$ for some polynomials $q(x)$ and $r(x)$ where either $r(x) = 0$ or $\deg r(x) < \deg b(x)$. ✋☺

The polynomial q(x) is called the **quotient** and r(x) is called the **remainder**.

If the remainder on dividing a(x) by b(x) is zero we say that b(x) **divides** a(x), or that a(x) is a **multiple** of b(x). If we can't be bothered saying it in words we just write b(x) $|$ a(x) and read it as "b(x) divides a(x)".

Example 9: Since $x^2 - 5x + 6 = (x - 2)(x - 3)$ it is true that $x - 2 \mid x^2 - 5x + 6$.
But $x^2 - 2x + 7$ does not divide $2x^3 + 5x - 3$.

§9.5. Greatest Common Divisors

A **greatest common divisor** (or **GCD**) of a(x) and b(x) is a polynomial of largest degree which divides them both.

Example10: The polynomial $x - 1$ divides both $x^2 - 1$ and $x^2 - 2x + 1$. No polynomial of higher degree divides both so $x - 1$ is a greatest common divisor.

But $2x - 2$ and $\frac{1}{4}x - \frac{1}{4}$ divide them both and they have the same degree as $x - 1$. What distinguishes $x - 1$ from these others is that it is monic — its leading coefficient is 1.

We would like to be able to define *the* GCD to be the monic one. But isn't it conceivable that there's more than one?

Imagine that we had two polynomials a(x) and b(x). Suppose also that both are divisible by x – 2, and both are divisible by x + 5, and that no quadratic, or polynomial of higher degree divides both a(x) and b(x). If such a situation was possible, what would be the GCD. of a(x) and b(x)? Would the GCD be x – 2 or would it be x + 5? Both would be common divisors of largest degree that divide both – and both are monic.

The answer is that such a situation cannot arise. If x – 2 and x + 5 were both common divisors, their product (x – 2)(x + 5) would also have to be a common divisor. But the product of two divisors is not always a divisor. For example both $x^2 - 1$ and $x^2 - 2x + 1$ are divisors of the cubic $(x - 1)^3(x + 1)$ but their product, $(x^2 - 1)(x^2 - 2x + 1)$, has degree 4 and so can't possibly be a divisor of the cubic. To help clarify this situation we show that there is another characterisation of GCDs.

Theorem 4: Let S be the set of all polynomials which are expressible in the form:

$$a(x)h(x) + b(x)k(x).$$

Then any polynomial in S of lowest degree is a greatest common divisor of a(x) and b(x).

Proof: S includes, of course, the zero polynomial, where h(x) = k(x) = 0, and it includes both a(x) and b(x) themselves. As well it includes polynomials such as $a(x)(x^2 + 1) - b(x)x^3$ which generally have higher degree than $a(x)$ and $b(x)$ themselves but which, because of cancellation, could have lower degree.

Choose a polynomial $m(x)$ from S with lowest degree.

Coming from S, $m(x) = a(x)h(x) + b(x)k(x)$ for some polynomials $h(x)$ and $k(x)$. We now show that $m(x)$ is a common divisor of $a(x)$ and $b(x)$.

Let $a(x) = m(x)q(x) + r(x)$ where $r(x) = 0$ or has lower degree than $m(x)$. Making $r(x)$ the subject of the formula we get:

$$r(x) = a(x) - m(x)q(x) = a(x)(1 - h(x)q(x)) + b(x)(-k(x)q(x)).$$

This means that $r(x)$ belongs to S, but if $r(x) = 0$ this would contradict the fact that $m(x)$ had lowest degree of any element of S. Hence $r(x) = 0$ and so $m(x)$ divides a(x). Similarly $m(x)$ divides $b(x)$ and so it is a common divisor.

But is $m(x)$ a *greatest* common divisor? Can there be a common divisor having larger degree than that of $m(x)$?

Suppose that $d(x)$ is any common divisor. Since $d(x)$ divides both $a(x)$ and $b(x)$, $d(x)$ must divide $m(x) = a(x)h(x) + b(x)k(x)$. This means that $\deg d(x) \le \deg m(x)$. So no common divisor can have larger degree than $m(x)$ and so m(x) is a *greatest* common divisor. ✋☺

Theorem 5: All GCDs of a given pair of polynomials are just constant multiples of one another. There is just one monic one among them.

Proof: Let $m(x) = a(x)h(x) + b(x)k(x)$ be the G.C.D. of $a(x)$ and $b(x)$ obtained as above. Let $d(x)$ be any G.C.D. of $a(x)$ and $b(x)$. Since $d(x)$ divides both $a(x)$ and $b(x)$ it must divide $m(x)$. So deg $d(x) \le$ deg $m(x)$. But since $d(x)$ is a greatest common divisor, we must have deg $d(x) =$ deg $m(x)$.

Now for one polynomial to divide another of the same degree, it must be that they are just constant multiples of one another. Hence all GCDs are constant multiples of $m(x)$ and hence of each other. ✋☺

Example 11: The GCD of $x^2 - 1$ and $x^2 - 2x + 1$ is $x - 1$. The above theorem shows that we can express $x - 1$ in the form $(x^2 - 1)h(x) + (x^2 - 2x + 1)k(x)$. Clearly the x^2 terms have to cancel out. A suitable expression is $x - 1 = (x^2 - 1)(½) + (x^2 - 2x + 1)(-½)$.
Here the $h(x)$ and $k(x)$ are constant polynomials, but in more complicated cases they would have higher degree.

Theorem 5: For all a, b, q (either integers or polynomials)

$$GCD(a, b) = GCD(a - bq, b).$$

Proof: Let $d_1 = GCD(a, b)$ and $d_2 = GCD(a - bq, b)$.
Then $d_1 \mid a - bq$ and $d_1 \mid b$ so $d_1 \mid d_2$. And since $a = (a - bq) + bq$, $d_2 \mid a$ and $d_2 \mid b$ so $d_2 \mid d_1$ and $d_2 \mid d_1$. Since d_1, d_2 are both positive (in the case of integers) or both monic (in the case of polynomials), $d_1 = d_2$. ✋☺

§9.6. The Euclidean Algorithm

You may recall the Euclidean Algorithm as it applies to finding the GCD of two integers. The same process can be easily adapted to polynomials. It's just that the algebra, when working with polynomials, can get a little messier than the arithmetic, when working with integers.

> **THE EUCLIDEAN ALGORITHM**
> **To find GCD(f(x), g(x)):**
> Let c(x) be the remainder on dividing f(x) by g(x).
> Replace f(x) by g(x).
> Replace g(x) by c(x).
> Continue from the top until g(x) $\neq$ 0.
> When g(x) = 0, make the last non-zero remainder monic.
> This is the required GCD.

Example 12: Find the G.C.D. of the polynomials $x^3 - 1$ and $x^4 + x^3 - x^2 + x - 2$.

Solution: We divide the cubic into the quartic. (Doing it the other way round wouldn't be wrong but all the first division would achieve would be to swap them.)

$$\begin{array}{r|l} & x + 1 \\ x^3 - 1 & x^4 + x^3 - x^2 + x - 2 \\ & \underline{x^4 \qquad\qquad - x} \\ & x^3 - x^2 + 2x - 2 \\ & \underline{x^3 \qquad\qquad - 1} \\ & -x^2 + 2x - 1 \end{array}$$

Since we are going to make the answer monic at the end we are entitled to multiply or divide any remainder by any non-zero constant. So we will replace our remainder by $x^2 - 2x + 1$.

$$
\begin{array}{r}
x + 2 \\
x^2 - 2x + 1 \overline{)\, x^3 \qquad\qquad\qquad - 1} \\
\underline{x^3 - 2x^2 \quad + x} \\
2x^2 \quad - x - 1 \\
\underline{2x^2 \quad - 4x + 2} \\
3x - 3
\end{array}
$$

We replace $3x - 3$ by $x - 1$.

$$
\begin{array}{r}
x \;\; - 1 \\
x - 1 \overline{)\, x^2 - 2x + 1} \\
\underline{x^2 - \;x} \\
-x \;\; + 1 \\
\underline{-x \;\; + 1} \\
0
\end{array}
$$

The last non-zero remainder is $x - 1$. This is the required GCD.

Example 13: Find the G.C.D. of the polynomials $x^3 + 1$ and $x^4 + x^3 - x^2 + x$.

Solution:

$$\begin{array}{r l}
 & \qquad\qquad\qquad x + 1 \\ \cline{2-2}
x^3 + 1 \,) & x^4 + x^3 - x^2 + x \\
 & \underline{x^4 \qquad\qquad\quad + x} \\
 & \qquad x^3 - x^2 \\
 & \qquad \underline{x^3 \qquad\qquad + 1} \\
 & \qquad\quad - x^2 \qquad - 1
\end{array}$$

We change this remainder to $x^2 + 1$.

$$\begin{array}{r l}
 & \qquad\quad x \\ \cline{2-2}
x^2 + 1 \,) & x^3 \qquad\quad + 1 \\
 & \underline{x^3 \quad + x \quad} \\
 & \qquad - x + 1
\end{array}$$

We change this to $x - 1$.

$$\begin{array}{r l}
 & \qquad\quad x + 1 \\ \cline{2-2}
x - 1 \,) & x^2 \qquad\quad + 1 \\
 & \underline{x^2 - x \qquad} \\
 & \qquad\quad x + 1 \\
 & \qquad\quad \underline{x - 1} \\
 & \qquad\qquad\quad 2
\end{array}$$

The last non-zero remainder (made monic) is 1 so the GCD of these two polynomials is 1. We say that the polynomials have “no common factor” (which is not strictly true — we actually mean that there is no common factor other than constants), or that they are **coprime**.

§9.7. Substitution and the Remainder Theorem

If f(x) $\in$ F[x], in other words f(x) is a polynomial in x with coefficients coming from the field F, and $\alpha \in$ F, we define $f(\alpha)$ to be the number, in F, that results from replacing, or **substituting**, x in the polynomial by the value α.

Example 14: If $f(x) = x^2 + x - 2$ then $f(2) = 4 + 2 - 2 = 4$, $f(0) = -2$ and $f(1)=0$.

The following theorem connects the ideas of substitution and remainder.

Theorem 6 (Remainder Theorem): The remainder on dividing f(x) by x – α is f(α).

Proof: By the Division Algorithm, $f(x) = (x - \alpha)q(x) + r(x)$ for some polynomials q(x), r(x) and the remainder r(x) is either zero or has degree less than 1. In other words r(x) must be a constant polynomial, so we can drop the (x) and just call it r.

Now substituting $x = \alpha$ into the equation $f(x) = (x - \alpha)q(x) + r$, we get $f(\alpha) = r$. ✋☺

Corollary: The polynomial f(x) is divisible by x – α if and only if $f(\alpha) = 0$.

Example 15: We saw that if $f(x) = x^2 + x - 2$ then $f(2) = 4$. The Remainder Theorem concludes that 4 must be the remainder on dividing f(x) by x – 2.

Numbers which produce zero when substituted into a polynomial $f(x)$ are just the solutions of the polynomial equation $f(x) = 0$. They're called the **zeros** of the polynomial and they're quite important features of the polynomial.

§9.8. Zeros of Polynomials

A **zero** (or 'root') of a polynomial $f(x)$ is a number, α, such that $f(\alpha)=0$. Solving a polynomial equation $f(x) = 0$ therefore means finding all its zeros.

But where do we look for potential zeros? From the coefficient field? But here we have to be a little bit careful. Does the polynomial $f(x) = x^2 + 1$ have any zeros? That depends. The coefficients are real numbers so we could consider $f(x)$ as belonging to the set of real polynomials, $\mathbb{R}[x]$. If so, there are no zeros. But we can just as validly consider $f(x)$ as belonging to the set of complex polynomials $\mathbb{C}[x]$ (remember the set $\mathbb{C}$ includes all of the real numbers). Then we'd have two zeros for $f(x)$, namely $\pm$ i. Frequently we switch from one field to another. So we can say that $x^2 + 1$ has no real zeros, but two complex ones.

The corollary to the Remainder Theorem can be expressed by saying that the number α is a root of $f(\alpha)$ if and only if $x - \alpha$ is one of its factors.

Now the polynomial $x - \alpha$ has degree 1 and it is called a **linear** polynomial. So there's a connection between linear factors and zeros of a polynomial.

Theorem 7: A polynomial has a zero if and only if it has a linear factor.
Proof: Whenever we have a zero, α, we have a linear factor $x - \alpha$. Conversely having a linear factor $bx + c$ for a polynomial means that we have a zero, $x = -c/b$. ✋☺

If we know one zerot of a polynomial we can use the remainder theorem and divide by the corresponding linear factor. The other zeros will then be zeros of the quotient.

Example 16: Given that $x = 2$ is a zero of $x^3 - x^2 - x - 2$, find the other two zeros.
Solution: By the remainder theorem $x - 2$ is a factor. Dividing the cubic by $x - 2$ we get the other factor which we proceed to solve.
So $x^3 - x^2 - x - 2 = (x - 2)(x^2 + x + 1)$. The other zeros are those of $x^2 + x + 1$, viz $\frac{1}{2}(-1 \pm \sqrt{3}i)$.

§9.9. Multiplicity of Zeros

The **multiplicity** of a zero α of a polynomial $f(x)$ is the largest k such that $(x - \alpha)^k$ divides $f(x)$.
A **multiple** (or repeated) zero is one whose multiplicity is at least 2.

Theorem 8: Suppose that α has multiplicity k in g(x) = GCD[f(x), f $'(x)$].
Then α has multiplicity k + 1 in f(x).
Proof: Suppose α has multiplicity m in f(x).
Then $f(x) = (x - \alpha)^m h(x)$ for some h(x) where $h(\alpha) \neq 0$.
Now $f'(x) = m(x - \alpha)^{m-1}h(x) + (x - \alpha)^m h'(x)$.
Now $(x - \alpha)^{m-1}$ divides g(x). If $(x - \alpha)^m$ divides g(x) then $x - \alpha$ divides h(x), a contradiction.
Hence α is a zero of g(x) with multiplicity m – 1. ✋☺

In fact it can be shown that the multiplicity of a zero is the largest k such that α is a common zero of f(x) and all its derivatives up to the $f^{(k-1)}(x)$, the (k – 1)st derivative.

Example 17: Let $f(x) = x^4 - 5x^3 + 9x^2 - 7x + 2$.
Then $f'(x) = 4x^3 - 15x^2 + 18x - 7$.

$$
\begin{array}{rl}
 & x - 5/4 \\
4x^3 - 15x^2 + 18x - 7\ ) & 4x^4 - 20x^3 + 36x^2 - 28x + 8 \\
 & \underline{4x^4 - 15x^3 + 18x^2 - 21x + 8} \\
 & -\ 5x^3 + 18x^2 - 21x + 8 \\
 & \underline{-5x^3 + 75/4x^2 - 45/2x + 35/4} \\
 & -3/4x^2 + 3/2x - 3/4
\end{array}
$$

$$
\begin{array}{rl}
 & 4x - 7 \\
x^2 - 2x + 1\,) & 4x^3 - 15x^2 + 18x - 7 \\
 & \underline{4x^3 - 8x^2 + 4x} \\
 & -7x^2 + 14x - 7 \\
 & \underline{-7x^2 + 14x - 7} \\
 & 0
\end{array}
$$

Notice how at certain stages I multiplied by a constant to reduce the number of fractions.
The last non-zero remainder is the GCD, namely $x^2 - 2x + 1 = (x - 1)^2$. Hence 1 has multiplicity 3 in f(x).

§9.10. Sum and Product of Zeros

If α and β are the zeros of the quadratic $ax^2 + bx + c$ then:

$$\alpha + \beta = -\frac{b}{a} \text{ (sum of the roots) and}$$

$$\alpha\beta = \frac{c}{a} \text{ (product of the roots).}$$

Example 18: If α and β are the zeros of $x^2 + 3x + 7$, find:
(i) $\alpha + \beta$; (ii) $\alpha\beta$; (iii)$\alpha^2 + \beta^2$; (iv) $\frac{1}{\alpha} + \frac{1}{\beta}$.

Solution: (i) $\alpha + \beta = -3$; (ii) $\alpha\beta = 7$; (iii)$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = 9 - 14 = -5$ (if you are worried by getting a negative sum of squares just remember that all this means is that the roots are non-real) ; (iv) $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta} = -\frac{3}{7}$.

Example 19: If α and β are the roots of $2x^2 - 5x + 1$, find (i) $\alpha^3 + \beta^3$; (ii) $\sqrt{\alpha} + \sqrt{\beta}$.

Solution: $\alpha + \beta = 5/2$ and $\alpha\beta = ½$. We must now express each expression in terms of these.

(i) $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha^2\beta - 3\alpha\beta^2 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta) = \frac{125}{8} - \frac{15}{4} = \frac{95}{8}$.

(ii) $(\sqrt{\alpha} + \sqrt{\beta})^2 = \alpha + \beta + 2\sqrt{\alpha\beta} = \frac{5}{2} + \frac{2}{\sqrt{2}} = \frac{5 + 2\sqrt{2}}{2}$. Hence $\sqrt{\alpha} + \sqrt{\beta} = \sqrt{\frac{5 + 2\sqrt{2}}{2}}$.

(The quadratic does have positive real roots and we are taking positive square roots throughout.)

For the zeros α, β, γ of the cubic $ax^3 + bx^2 + cx + d$ we get similar results, but there's an additional combination of the zeros we need to consider.

$\alpha + \beta + \gamma = -\frac{b}{a}$ (sum of the zeros);

$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$ (sum of the zeros taken two at a time);

$\alpha\beta\gamma = -\frac{d}{a}$ (product of the zeros).

Example 20: If α, β, γ are the zeros of the cubic $x^3 + 5x^2 - 2x - 3$, find the value of $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$.

Solution: $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\alpha\beta + \beta\gamma + \gamma\alpha}{\alpha\beta\gamma} = -\frac{2}{3}$.

The three expressions $\alpha + \beta + \gamma$, $\alpha\beta + \beta\gamma + \gamma\alpha$ and $\alpha\beta\gamma$ are **symmetric** in α, β and γ. This means that α, β and γ can be permuted in any order and the value of these expressions is unchanged. They're called the **elementary symmetric functions** on 3 variables since every symmetric function on α, β, γ can be expressed in terms of them. We denote these elementary symmetric finctions by: $\Sigma\alpha$, $\Sigma\alpha\beta$ and $\Sigma\alpha\beta\gamma$, although we would probably write $\Sigma\alpha\beta\gamma$ more simply as $\alpha\beta\gamma$.

However if we had four zeros: α, β, γ, δ for a quartic, $\Sigma\alpha\beta\gamma$ would denote $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta$.

Example 21: Express the following symmetric functions in terms of the elementary symmetric ones:

(i) $\frac{\alpha\beta}{\gamma} + \frac{\beta\gamma}{\alpha} + \frac{\gamma\alpha}{\beta}$;

(ii) $\alpha^2\beta + \beta^2\gamma + \gamma^2\alpha + \alpha\beta^2 + \beta\gamma^2 + \gamma\alpha^2$;

(iii) $\alpha^3 + \beta^3 + \gamma^3$.

Solution:

(i) $$\frac{\alpha\beta}{\gamma} + \frac{\beta\gamma}{\alpha} + \frac{\gamma\alpha}{\beta} = \frac{\alpha^2\beta^2 + \beta^2\gamma^2 + \alpha^2\gamma^2}{\alpha\beta\gamma}$$

$$= \frac{(\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2\alpha^2\beta\gamma - 2\alpha\beta^2\gamma - 2\alpha\beta\gamma^2}{\alpha\beta\gamma}$$

$$= \frac{(\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2\alpha\beta\gamma(\alpha + \beta + \gamma)}{\alpha\beta\gamma}.$$

(ii) $\alpha^2\beta + \beta^2\gamma + \gamma^2\alpha + \alpha\beta^2 + \beta\gamma^2 + \gamma\alpha^2$

$$= (\alpha + \beta + \gamma)(\alpha\beta + \beta\gamma + \gamma\alpha) - 3\alpha\beta\gamma.$$

(iii) $\alpha^3 + \beta^3 + \gamma^3$

$= (\alpha + \beta + \gamma)^3 - 3\alpha^2\beta - 3\beta^2\gamma - 3\gamma^2\alpha$

$\qquad - 3\alpha\beta^2 - 3\beta\gamma^2 - 3\gamma\alpha^2 - 6\alpha\beta\gamma$

$= (\alpha + \beta + \gamma)^3 - 3(\alpha + \beta + \gamma)(\alpha\beta + \beta\gamma + \gamma\alpha) + 3\alpha\beta\gamma.$

The above results extend to polynomials of higher degree. If $f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1\text{x} + a_0$ is a polynomial of degree n with zeros $\alpha_1, \alpha_2, \ldots, \alpha_n$ then $\Sigma\alpha_1\,\alpha_2\ldots\alpha_r$, the sum of all products of zeros taken r at a time, is $(-1)^r\dfrac{a_{n-r}}{a_n}$.

Moreover every symmetric function in the α_i can be expressed in terms of these elementary symmetric functions.

Example 22: If $\alpha, \beta, \gamma, \delta$ are the zeros of

$$x^4 + 5x^3 - 3x^2 - 3x - 7$$

find $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$.

Solution: $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$

$= (\alpha + \beta + \gamma + \delta)^2 - 2(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)$

$= \Sigma\alpha - 2\Sigma\alpha\beta = -5 - 2(-3) = 1.$

§9.11. Real and Complex Polynomials

A real polynomial can be graphed in the usual way and real zeros correspond to places where the curve cuts the x-axis. A real polynomial has at least one real zero if its graph crosses the x-axis.

Theorem 9: Real polynomials of odd degree have at least one real zero.

Proof: If a(x) has odd degree, and the leading coefficient is positive, then a(x) $\to -\infty$ as $x \to -\infty$ and a(x) $\to \infty$ as $x \to \infty$. (If the leading coefficient is negative these are reversed.) It therefore takes both positive and negative values and since it moves continuously between positive and negative values it must cross the x-axis in at least one place.

Real polynomials of even degree can have real zeros too, but there are plenty which don't (such as $x^2 + 1$).

We can't graph a polynomial with complex coefficients without using 4 dimensions. However a very important theorem, easy to state but not so easy to prove, states that *every* complex polynomial (apart from constants) has complex zeros (some or all of which could be real). This is called the Fundamental Theorem of Algebra.

Theorem 10 (Fundamental Theorem of Algebra):
If $f(x) \in \mathbb{C}[x]$ and $\deg f(x) \geq 1$, then $f(\alpha) = 0$ for some $\alpha \in \mathbb{C}$.✋

The Fundamental Theorem states that not only complex quadratics but complex cubics, complex quartics, ... have complex zeros. Because of this we say that **C** is **algebraically closed**. No new numbers need to be invented.

The history of number consisted of mathematicians continuing to extend the number system so as to provide zeros for polynomials which would otherwise have none.

God, it has been said, gave us the natural numbers 1,2,3,... and man created the rest. In fact even the counting numbers were invented by man, because they are abstract concepts. Or one might argued that God invented all numbers, and man has merely discovered them! Much energy was expended on whether certain numbers exist, and the term ‘imaginary number’ stems from those discussions. The modern view is that all numbers exist if we choose for them to exist provided, of course, that they don’t lead us into a contradiction. Defining ∞ to be 1/0 is no good because it would cause the number system to collapse. (If $\infty = 1/0$ then $0\infty = 1 = (0 + 0)\infty = 1 + 1 = 2$.)

When we just had the natural numbers 1,2,3,... we had to say that the equation $2x = 1$ has no solutions. So fractions were invented. Then $2x = 1$ could be solved but $x^2 = 2$ couldn’t. So along came the irrational numbers. Since $\sqrt{2}$ now exists, $x^2 = 2$ now has a solution (just one because so far there was no such thing as a negative number).

This all happened back in classical times. But despite all these extra numbers that had been invented, an equation such as $x + 2 = 1$ still had no solution. Negative numbers took another thousand years or so arrive. Once they did then $x + 2 = 1$ was OK, but $x^2 + 1 = 0$ could not be solved. The last stage was the extension from the real numbers to the complex numbers which took place in the 17th century. Now a polynomial equation such as $x^2 + 1 = 0$ could be solved, but what about $x^2 + i = 0$, or $x^5 + ix^3 = 1 + 2i$? Might we not have to go on inventing more and more numbers? Might not our number system spring new leaks whenever we try to patch up existing ones?

After all, whenever we extend our number system we introduce new equations. But the marvellous thing is that having arrived at the system of complex numbers we had, in a sense, reached perfection. No new numbers are necessary. We can solve *all* polynomial equations, with complex coefficients, entirely from within the complex number system itself. There is a complex number x such that $x^2 + i = 0$ (in fact two of them). Furthermopre there's a complex number x such that $x^5 + ix^3 = 1 + 2i$ (in fact five such solutions). The Fundamental Theorem shows that we needn't extend our number system any further.

As a result of the Fundamental Theorem of Algebra every polynomial with complex coefficients can be factorised completely into linear factors. Having found one zero α we can divide by $x - \alpha$ and apply the Fundamental Theorem of Algebra on the quotient to find a second zero, and so on, until we reach a constant polynomial, If the coefficients are real, and we insist that the factors have real coefficients, we can't expect to be able to factorise the polynomial quite so completely. However we can factorise any real polynomial into real factors which, at worst, are quadratic. This comes about because the non-real roots of a real polynomial split into conjugate pairs.

Theorem 11 (Conjugate Pairs):
The non-real zeros of a real polynomial come in conjugate pairs.
Proof: Suppose α is a zero of the real polynomial

$$\mathrm{a}(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0.$$

Then $\mathrm{a}(\alpha) = 0$. Hence $a_n\alpha^n + a_{n-1}\alpha^{n-1} + \ldots + a_1\alpha + a_0 = 0$.
Taking conjugates of both sides we see that the conjugate of α is also a zero. ✋☺

Example 23: The five zeros of $f(x) = x^5 - 2x^4 + 6x^3 - 2x^2 + 5x$ are 0, $1 \pm 2i$ and $\pm$ i.
The factorisation of $f(x)$ over **C** is $f(x) = x(x-i)(x+i)(x-1-2i)(x+1+2i)$.
Grouping the linear factors which correspond to conjugate zeros this becomes
$f(x) = x(x^2 + 1)(x^2 - 2x + 5)$, a factorisation over **R**.

Theorem 11 can be useful in factorising certain real polynomials if we're given a non-real zero.

Example 24: Assuming that 2 + i is a zero of
$a(x) = x^4 - 6x^3 - 22x^2 + 130x - 175$
factorise it completely over the rationals.

Solution: Since the coefficients are real, 2 – i is also a zero and hence
$(x - 2 - i)(x - 2 + i) = x^2 - 4x + 5$ is a factor.
Dividing by this quadratic we get the other factor: $x^2 - 2x - 35$.
This factorises as $(x - 7)(x + 5)$. Hence $a(x) = (x^2 - 4x + 5)(x - 7)(x + 5)$ is the complete factorisation over the rationals.

EXERCISES FOR CHAPTER 9

Exercise 1: Find the degree, the leading coefficient and the zeros of the polynomial
$f(x) = 5x^6 - 6x^5 - x^7$.

Exercise 2:
Which of the following are polynomials in x?

(a) $x^5 + x$;
(b) $x + x^{-1}$;
(c) $1 + x^2 + x^4 + ...$;
(d) 42;
(e) $x^2 + \sqrt{x} + 1$.

Exercise 3: Find the degree of the polynomial
$f(x) = (2^n - 8)x^{2n+1} + 4x^5 + x^2 + 7$ in terms of n.

Exercise 4: Find the quotient and remainder on dividing $f(x) = x^4 + 7x + 2$ by $x - 3$.

Exercise 5: Find the remainder on dividing $x^7 + x^4 + x^2 + 1$ by $x^2 - 3$.

Exercise 6: Show that $x - 3$ is a factor of the cubic $f(x) = x^3 - 7x^2 + 20x - 24$. Hence find the zeros of $f(x)$.

Exercise 7: Given that $1 + i$ is a zero of the polynomial $x^4 - 4x^3 + 23x^2 - 38x + 34$, find its other three zeros.

Exercise 8: If α and β are the zeros of $4x^2 - 12x + 7$ find the values of:

(i) $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha}$;

(ii) $\alpha^2\beta + \alpha\beta^2$;

(iii) $(\alpha - \beta)^2$;

(iv) $\alpha - \beta$.

Exercise 9: If α, β and γ are the zeros of the cubic $f(x) = x^3 - 7x^2 + 11x - 4$, find the value of:

$$E = (\alpha + \beta)(\beta + \gamma)(\gamma + \alpha).$$

Exercise 10: Find the remainder on dividing $f(x) = x^{13} + 7x^6 - 5x^2 + x - 2$ by $x - 1$.

Exercise 11: Find the remainder on dividing $x^5 - 2x^3 + 5x^2 - 7$ by $x^2 + x + 2$.

Exercise 12: Given that $x = 2$ is a zero of $x^3 - 6x^2 + 9x - 2$, find the other two zeros.

Exercise 13: Given that $x = \mathrm{i}$ is a zero of $6x^4 - x^3 + 4x^2 - x - 2$, find the other three zeros.

Exercise 14: If α and β are the zeros of the quadratic $2x^2 - 3x + 8$, find the values of:

(i) $\alpha^3 + \beta^3$;

(ii) $\dfrac{1}{\alpha^3} + \dfrac{1}{\beta^3}$;

(iii) $\alpha\beta^4 + \beta\alpha^4$.

Exercise 15: If α, β, γ are the zeros of $x^3 + 5x^2 + 2x - 3$ find the values of:

(i) $\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2$;

(ii) $\alpha^2 + \beta^2 + \gamma^2$;

(iii) $\dfrac{\alpha}{\beta\gamma} + \dfrac{\beta}{\gamma\alpha} + \dfrac{\gamma}{\alpha\beta}$.

Exercise 16: If $f(x) = x^9 + (k + 1)x^8 + (k^2 + k + 1)x^7 + \ldots$ find the sum of the squares of the zeros of $f(x)$ and hence show that $f(x)$ has at least one non-real zero.

Exercise 17: Find the GCD of $2x^4 + 7x^3 + 2x^2 + 9x + 7$ and $2x^3 + 9x^2 + 9x + 7$.

Exercise 18: The cubic $f(x) = x^3 - 13x^2 + 56x - 80$ has a repeated zero. Find it.

Exercise 19: Find the GCD of $a(x) = 10x^2 + x - 21$ and $b(x) = 8x^2 + 30x + 27$.

Exercise 20: Find the GCD of $x^5 + 1$ and $x^4 + 1$.

Exercise 21: Find any multiple zeros of the polynomial $f(x) = x^4 + 3x^3 + 4x^2 + 3x + 1$.

Exercise 22: Show that the polynomial

$$f(x) = x^4 - 2x^3 + 5x^2 - 4x + 4$$

is a perfect square (i.e. the square of a polynomial). [HINT: Look for multiple zeros.]

SOLUTIONS FOR CHAPTER 9

Exercise 1: $f(x) = -x^7 + 5x^6 - 6x^5$
$= -x^5(x-2)(x-3)$, so $\deg f(x) = 7$; the leading coefficient is -1 and the zeros are 0, 2, 3.

Exercise 2: (a) and (d) are polynomials; the others are not.

Exercise 3: For $n = 1$, $f(x) = -6x^3 + 4x^5 + x^2 + 7$ and so $\deg f(x) = 5$.
For $n = 2$, $f(x) = -4x^5 + 4x^5 + x^2 + 7 = x^2 + 7$ so
$\deg f(x) = 2$.
For $n = 3$, $f(x) = 0x^7 + 4x^5 + x^2 + 7$ and so $\deg f(x) = 5$.
For $n \geq 4$, $\deg f(x) = 2n+1$.

Exercise 4:

$$
\begin{array}{rrrrrr}
 & & x^3 & +3x^2 & +9x & +34 \\ \cline{2-6}
x-3\,) & x^4 & & & +7x & +2 \\
 & \underline{x^4} & \underline{-3x^3} & & & \\
 & & 3x^3 & & +7x & +2 \\
 & & \underline{3x^3} & \underline{-9x^2} & & \\
 & & & 9x^2 & +7x & +2 \\
 & & & \underline{9x^2} & \underline{-27x} & \\
 & & & & 34x & +2 \\
 & & & & \underline{34x} & \underline{-102} \\
 & & & & & 104
\end{array}
$$

Hence the quotient is $x^3 + 3x^2 + 9x + 34$ and the remainder is 104. If we just wanted the reminder it would be much easier to use the Remainder Theorem.
The remainder is f(3) = 81 + 21 + 2 = 104.

Exercise 5:

$$
\begin{array}{r l}
 & x^5 \qquad + 3x^3 + x^2 + 9x + 4 \\
\cline{2-2}
x^2 - 3 \,) & x^7 + \qquad x^4 \qquad + x^2 \qquad + 1 \\
 & \underline{x^7 - 3x^5} \\
 & \qquad 3x^5 + x^4 \qquad + x^2 \qquad + 1 \\
 & \qquad \underline{3x^5 \qquad - 9x^3} \\
 & \qquad\qquad x^4 + 9x^3 + x^2 \qquad + 1 \\
 & \qquad\qquad \underline{x^4 \qquad - 3x^2} \\
 & \qquad\qquad\qquad 9x^3 + 4x^2 \qquad + 1 \\
 & \qquad\qquad\qquad \underline{9x^3 \qquad - 27x} \\
 & \qquad\qquad\qquad\qquad 4x^2 + 27x + 1 \\
 & \qquad\qquad\qquad\qquad \underline{4x^2 \qquad - 12} \\
 & \qquad\qquad\qquad\qquad\qquad 27x + 13
\end{array}
$$

The remainder is $27x + 13$.

Exercise 6: We could verify the fact that $x - 3$ is a factor by checking that f(3) = 0. However, since we will need to find the quotient we must carry out the long division process.

$$
\begin{array}{r l}
 & \quad\; x^2 \quad - 4x \quad + 4 \\
\cline{2-2}
x - 3 \,) & x^3 - 7x^2 + 20x - 24 \\
 & \underline{x^3 - 3x^2 \qquad\qquad\quad} \\
 & \qquad - 4x^2 + 20x - 24 \\
 & \qquad \underline{- 4x^2 + 12x \qquad\;} \\
 & \qquad\qquad\qquad 8x - 24 \\
 & \qquad\qquad\qquad \underline{8x - 24} \\
 & \qquad\qquad\qquad\qquad 0
\end{array}
$$

The quotient is $x^2 - 4x + 8$.
Solving this quadratic we get $x = 2 \pm 2i$.
So the zeros of f(x) are 3, $2 \pm 2i$.

Exercise 7: Since the polynomial has real coefficients, 1 – i the conjugate of 1 + i, must also be a zero.
Hence $x - (1 + i)$ and $x - (1 - i)$ are factors.
Thus $(x - 1 - i)(x - 1 + i) = x^2 - 2x + 2$ is a factor.
We now divide this into the quartic to obtain the other quadratic factor.

$$
\begin{array}{r l}
 & \qquad\qquad\qquad x^2 \; - 2x + 17 \\
\cline{2-2}
x^2 - 2x + 2 \,) & x^4 - 4x^3 + 23x^2 - 38x + 34 \\
 & \underline{x^4 - 2x^3 \quad + 2x^2 \qquad\qquad\quad} \\
 & \qquad - 2x^3 + 21x^2 - 38x + 34 \\
 & \qquad \underline{- 2x^3 \quad + 4x^2 \; - 4x \qquad} \\
 & \qquad\qquad\qquad 17x^2 - 34x + 34 \\
 & \qquad\qquad\qquad \underline{17x^2 - 34x + 34} \\
 & \qquad\qquad\qquad\qquad\qquad\quad 0
\end{array}
$$

Hence the polynomial factorises as:

$$(x^2 - 2x + 2)(x^2 - 2x + 17).$$

Solving $x^2 - 2x + 17 = 0$ gives $1 \pm 4i$. So the other three zeros are $1 - i$, $1 \pm 4i$.

Exercise 8: $\alpha + \beta = 3$, $\alpha\beta = 7/4$.

(i) $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha} = \dfrac{\alpha^2 + \beta^2}{\alpha\beta} = \dfrac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta} = \dfrac{22}{7}$;

(ii) $\alpha^2\beta + \alpha\beta^2 = \alpha\beta(\alpha + \beta) = 21/4$;

(iii) $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta = 9 - 7 = 2$;

(iv) $\alpha - \beta = \pm\sqrt{2}$.

Exercise 9: $\alpha + \beta + \gamma = 7$ etc.

Now rather than expand E note that

$$\alpha + \beta = (\alpha + \beta + \gamma) - \gamma = 7 - \gamma,$$

and similarly for the others. So $E = (7 - \alpha)(7 - \beta)(7 - \gamma)$. Now before you get carried away expanding this out, just note that this $f(x) = (x - \alpha)(x - \beta)(x - \gamma)$ and so E is simply $f(7) = 73$.

Exercise 10: The remainder is $f(1) = 2$.

Exercise 11:

$$
\begin{array}{r|llllll}
 & & & x^3 & - x^2 & - 3x & + 10 \\ \hline
x^2 + x + 2 & x^5 & & - 2x^3 & + 5x^2 & & - 7 \\
 & x^5 & + x^4 & + 2x^3 & & & \\ \cline{2-7}
 & & x^4 & - 4x^3 & + 5x^2 & & - 7 \\
 & & x^4 & - x^3 & - 2x^2 & & \\ \cline{3-7}
 & & & - 3x^3 & + 7x^2 & & - 7 \\
 & & & - 3x^3 & - 3x^2 & - 6x & \\ \cline{4-7}
\end{array}
$$

$$\begin{array}{r} 10x^2 \quad - 6x \quad - 7 \\ \underline{10x^2 + 10x + 20} \\ - 16x - 27 \end{array}$$

Exercise 12:

$$\begin{array}{r} x^2 - 4x + 1 \\ x - 2 \,\overline{)\, x^3 - 6x^2 + 9x - 2} \\ \underline{x^3 - 2x^2 \qquad\qquad} \\ - 4x^2 + 9x - 2 \\ \underline{- 4x^2 + 8x \quad} \\ x - 2 \\ \underline{x - 2} \\ 0 \end{array}$$

Exercise 13: Since i is a zero, so is –i.
Hence $(x - i)(x + i) = x^2 + 1$ is a factor.

$$\begin{array}{r} 6x^2 - x - 2 \\ x^2 + 1 \,\overline{)\, 6x^4 - x^3 + 4x^2 - x - 2} \\ \underline{6x^4 \qquad + 6x^2 \qquad\qquad} \\ - x^3 - 2x^2 - x - 2 \\ \underline{- x^3 \qquad - x \quad} \\ - 2x^2 \qquad - 2 \\ \underline{- 2x^2 \qquad - 2} \\ 0 \end{array}$$

Exercise 14: $\alpha + \beta = 3/2$, $\alpha\beta = 4$

(i) $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha^2\beta - 3\alpha\beta^2$

$$= \frac{9}{4} - 3\alpha\beta(\alpha + \beta)$$

$$= \frac{9}{4} - 12\left(\frac{3}{2}\right) = -\frac{63}{4}.$$

Exercise 15: $\alpha + \beta + \gamma = -5$, $\alpha\beta + \beta\gamma + \alpha\gamma = 2$, $\alpha\beta\gamma = 3$.

(i) $\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2 = \alpha\beta\gamma(\alpha + \beta + \gamma) = -15$.

(ii) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \alpha\gamma)$

$$= 25 - 4 = 21.$$

(iii) $\dfrac{\alpha}{\beta\gamma} + \dfrac{\beta}{\gamma\alpha} + \dfrac{\gamma}{\alpha\beta} = \dfrac{\alpha^2 + \beta^2 + \gamma^2}{\alpha\beta\gamma} = -\dfrac{21}{5}$.

Exercise 16: $\Sigma\alpha = -(k + 1)$ and $\Sigma\alpha\beta = k^2 + k + 1$.

Hence $\Sigma\alpha^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta = (k + 1)^2 - 2(k^2 + k + 1)$

$$= -(k^2 + 1).$$

If all the roots were real then $\Sigma\alpha^2 \geq 0$, whereas $-(k^2 + 1) < 0$.

Exercise 17:

$$\begin{array}{r|l} & \quad\quad\quad\quad\quad\quad\quad x - 1 \\ 2x^3 + 9x^2 + 9x + 7 & 2x^4 + 7x^3 + 2x^2 + 9x + 7 \\ & \underline{2x^4 + 9x^3 + 9x^2 + 7x \quad\quad} \\ & \quad - 2x^3 - 7x^2 + 2x + 7 \\ & \quad \underline{- 2x^3 - 9x^2 - 9x - 7} \\ & \quad\quad\quad\quad 2x^2 + 11x + 14 \end{array}$$

$$
\begin{array}{r l}
 & \underline{ \qquad\qquad\qquad\qquad\quad x - 1} \\
2x^2 + 11x + 14 &)\; 2x^3 + 9x^2 + 9x + 7 \\
 & \quad \underline{2x^3 + 11x^2 + 14x \phantom{{}+7}} \\
 & \qquad\quad - 2x^2 - 5x + 7 \\
 & \qquad\quad \underline{- 2x^2 - 11x - 14} \\
 & \qquad\qquad\qquad\quad 6x + 21
\end{array}
$$

At the end we’ll be making the last non-zero remainder monic, so at any stage we can multiply or divide by a non-zero constant to simplify the calculations.
So we’ use $2x + 7$ instead of $6x + 21$.

$$
\begin{array}{r l}
 & \underline{ \qquad\qquad x + 2} \\
2x + 7 &)\; 2x^2 + 11x + 14 \\
 & \quad \underline{2x^2 + 7x \phantom{{}+14}} \\
 & \qquad\qquad 4x + 14 \\
 & \qquad\qquad \underline{4x + 14} \\
 & \qquad\qquad\qquad\quad 0
\end{array}
$$

The last non-zero remainder, made monic, is $x + 7/2$.
This is the GCD.

Exercise 18: A repeated zero will be a zero of $f'(x) = 3x^2 - 26x + 56$ and so will be a zero of GCD(f(x), f $'$(x)).
For simplicity let’s replace f(x) by 3f(x).

$$
\begin{array}{r l}
 & \underline{ \qquad\qquad\qquad\qquad\quad x - 13/3} \\
3x^2 - 26x + 56 &)\; 3x^3 - 39x^2 + 168x - 240 \\
 & \quad \underline{3x^3 - 26x^2 + 56x \phantom{{}-240}} \\
 & \qquad\quad - 13x^2 + 112x - 240
\end{array}
$$

$$\begin{array}{r} \underline{-\,13x^2 \;+ 338/3x - 728/3} \\ -\,2/3x + \quad 8/3 \end{array}$$

Replace this remainder by the multiple
$x - 4 = (-3/2)(-2/3x + 8/3)$.

$$\begin{array}{r|l} & \quad\;\; 3x - 14 \\ \hline x - 4 & 3x^2 - 26x + 56 \\ & \underline{3x^2 - 12x \quad\;\;} \\ & \quad - 14x + 56 \\ & \quad \underline{- 14x + 56} \\ & \qquad\qquad 0 \end{array}$$

Exercise 19:

Rather than divide we can first subtract. This will simplify the arithmetic.

$$\begin{aligned} \mathrm{GCD}(a(x), b(x)) &= \mathrm{GCD}(a(x) - b(x), b(x)) \\ &= \mathrm{GCD}(2x^2 - 29x - 48, 8x^2 + 30x + 27). \\ &= \mathrm{GCD}(2x^2 - 29x - 48, (8x^2 + 30x + 27) \\ &\qquad\qquad - 4(2x^2 - 29x - 48)) \\ &= \mathrm{GCD}(2x^2 - 29x - 48, 146x + 219) \\ &= \mathrm{GCD}(2x^2 - 29x - 48, 2x + 3) \\ &\qquad \text{since } 146x + 219 = 73(2x + 3). \end{aligned}$$

$$\begin{array}{r|l} & \qquad\quad x - 16 \\ \hline 2x + 3 & 2x^2 - 29x - 48 \\ & \underline{2x^2 \;+\; 3x \quad\;\;} \\ & \quad - 32x - 48 \\ & \quad \underline{- 32x - 48} \\ & \qquad\qquad 0 \end{array}$$

Exercise 20:

$$
\begin{array}{r l}
 & \underline{\qquad\qquad x \qquad} \\
x^4 + 1\ ) & x^5 \qquad\qquad\quad + 1 \\
 & \underline{x^5 \qquad + x \qquad} \\
 & \qquad\quad - x \quad + 1
\end{array}
$$

Replace this remainder by $x - 1$.

$$
\begin{array}{r l}
 & \underline{\qquad x^3 + x^2 + x \qquad\qquad} \\
x - 1\ ) & x^4 \qquad\qquad\qquad\qquad + 1 \\
 & \underline{x^4 - x^3 \qquad\qquad\qquad\quad} \\
 & \qquad x^3 \qquad\qquad\quad + 1 \\
 & \qquad \underline{x^3 - x^2 \qquad\qquad\;} \\
 & \qquad\qquad x^2 \qquad + 1 \\
 & \qquad\qquad \underline{x^2 \qquad - 1} \\
 & \qquad\qquad\qquad\qquad\; 2
\end{array}
$$

Making this monic we see that the GCD is 1.

Exercise 21: $f'(x) = 4x^3 + 9x^2 + 8x + 3$.
For convenience replace $f(x)$ by by $4f(x)$.

$$
\begin{array}{r l}
 & \underline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad x + 3/4} \\
4x^3 + 9x^2 + 8x + 3\ ) & 4x^4 + 12x^3 \qquad + 16x^2 + 12x + 4 \\
 & \underline{4x^4 + 9x^3 \qquad\;\; + 8x^2 + 3x \qquad} \\
 & \qquad\quad 3x^3 \qquad + 8x^2 + 9x + 4 \\
 & \qquad\quad \underline{3x^3 + (27/4)x^2 + 6x + 9/4} \\
 & \qquad\qquad\qquad (5/4)x^2 + 3x + 7/4
\end{array}
$$

$$
\begin{array}{r l}
 & \underline{\qquad\qquad\qquad\qquad (4/5)x - 3/25} \\
5x^2 + 12x + 7\ ) & 4x^3 \qquad + 9x^2 \qquad + 8x \quad + 3 \\
 & \underline{4x^3 + (48/5)x^2 + (28/5)x \qquad} \\
 & \qquad\quad (-3/5)x^2 + (12/5)x + 3 \\
 & \qquad\quad \underline{(-3/5)x^2 - (36/25)x - 21/25} \\
 & \qquad\qquad\qquad\qquad (96/25)x + 96/25
\end{array}
$$

$$
\begin{array}{r l}
 & \underline{\qquad\quad 5x + 7} \\
x + 1\ ) & 5x^2 + 12x + 7 \\
 & \underline{5x^2 + 5x \qquad} \\
 & \qquad\quad 7x + 7 \\
 & \qquad\quad \underline{7x + 7} \\
 & \qquad\qquad\ 0
\end{array}
$$

Exercise 22: $f'(x) = 4x^3 - 6x^2 + 10x - 4$.

$$
\begin{array}{r l}
 & \underline{\qquad\qquad\qquad\qquad\qquad 0.5x - 0.25} \\
2x^3 - 3x^2 + 5x - 2) & x^4 - 2x^3 \quad + 5x^2 \quad - 4x + \quad 4 \\
 & \underline{x^4 - 1.5x^3 + 2.5x^2 \qquad - x \qquad\quad} \\
 & \quad -0.5x^3 + 2.5x^2 \quad - 3x \ + 4 \\
 & \quad \underline{-0.5x^3 + 0.75x^2 - 1.25x + 0.5} \\
 & \qquad\qquad\qquad 1.75x^2 - 1.75x + 3.5
\end{array}
$$

Replace this by $x^2 - x + 2$.

$$
\begin{array}{r l}
 & \underline{\qquad\qquad\qquad 2x - 1} \\
x^2 - x + 2\ ) & 2x^3 - 3x^2 + 5x - 2 \\
 & \underline{2x^3 - 2x^2 + 4x \qquad} \\
 & \qquad -x^2 \quad + x - 2 \\
 & \qquad \underline{-x^2 \quad + x - 2} \\
 & \qquad\qquad\qquad\quad 0
\end{array}
$$